## LITERATURE CITED

1. É. A. Gershbein, V. S. Shchelin, and S. A. Yunitskii, Kosm. Issled., 23, No. 3, 416-425 (1985).
2. H. K. Cheng, J. Astronaut. Sci., No. 92, 1-100 (1963).
3. R. V. Masek, D. Hender, and J. A. Forney, "Evaluation of aerodynamic uncertainties for the space shuttle," AIAA Paper No. 737, New York (1973).
4. É. A. Gershbein and V. G. Shcherbak, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 4, 134-142 (1987).
5. V. G. Shcherbak, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5, 143-150 (1987).
6. C. D. Scott, "Catalytic recombination of nitrogen and oxygen on high-temperature reusable insulation," AIAA Paper No. 1477, New York (1980).
7. V. V. Lunev, Hypersonic Aerodynamics [in Russian], Moscow (1975).


# COMPARATIVE EVALUATION OF APPROXIMATE METHODS FOR SOLVING ONE-DIMENSIONAL PROBLEMS INVOLVING MOVABLE BOUNDARIES


É. A. Bondarev and F. S. Popov



UDC 536.01



Estimates of accuracy have been obtained for the most extensively employed approximation methods for the solution of the Stefan problem.


The overwhelming majority of problems involving unknown movable boundaries (problems of the Stefan–Verigin type) are not solved in quadratures. It thus becomes necessary to construct rather simple approximate solutions which might be used not only in estimation calculations, but also to verify the effectiveness of computational algorithms based on finite-difference methods. An attempt is made in this article, through comparison of existing analytical solutions, to evaluate the accuracy and scope of applicability for the most popular approximate methods of solution for Stefan problems in the case of a plane-parallel flow of heat: the Leibenzon–Charnyi method (LChM) [1, 2], the Barenblatt–Goodman integral method (IM) [3, 4], the successive approximation method (SAM) [5], and the combination method (CM) whose essence is explained below.

In the general case, the parabolic equation

$$\frac{\partial T}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial T}{\partial x}\right) \tag{1}$$

is satisfied by the following functions [7]:

$$\Phi(\zeta) = \operatorname{erfc}(\zeta),\ \operatorname{Ei}(-\zeta^2),\ \frac{\exp(-\zeta^2)}{\zeta} - \sqrt{\pi}\operatorname{erfc}(\zeta) \tag{2}$$

respectively, for n = 0, 1, 2. Here

$$\zeta = x/2\sqrt{at}. \tag{3}$$

The solution of the specific boundary-value problem for Eq. (1) can be presented in the form

$$T = A + B\Phi(\zeta). \tag{4}$$

It is important here to underscore that A and B are not simply coefficients which must be determined from specific boundary-value problems, but rather integration constants. In par-


Institute of the Physicotechnical Problems of the North, Academy of Sciences of the USSR, Yakutsk.


Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 56, No. 2, pp. 302-306, February, 1989. Original article submitted December 7, 1987.

ticular, with a constant heat flow given at the movable boundary these coefficients will be functions of x and t, but in that case a solution of the form of (4) will not satisfy the heat conduction equation [6].

The coefficients A and B will be independent of x and t only under the following boundary and initial conditions:

$$T(0,\ t) = T_c,\ n = 0, \tag{5}$$

$$\lim_{x\to 0} (2x)^n \lambda_1 \frac{\partial T}{\partial x} = -Q(4a_1 t)^{(n-1)/2}. \tag{6}$$

For n = 1, 2, the left-hand side of Eq. (6) is multiplied by π.

$$T(x,\ 0) = T_0, \tag{7}$$

$$T(\infty,\ t) = T_0. \tag{8}$$

An additional two conditions are formulated at the phase separation boundary S(t) for the Stefan problem:

$$T_1 = T_2 = T_{\text{ph}}, \tag{9}$$

$$-\lambda_1 \frac{\partial T_1}{\partial x} + \lambda_2 \frac{\partial T_2}{\partial x} = l\rho\omega \frac{dS}{dt}. \tag{10}$$

In this case the coefficients A and B will be constant only if

$$S = 2\alpha\sqrt{a_1 t}. \tag{11}$$

We will carry out the comparative evaluation of the approximation methods on the example of the Stefan problem for plane-parallel heat flow (n = 0).

1. Boundary Condition of the 1st Kind. In this case the exact self-similar solution has the form [7]

$$\Theta_1 = \operatorname{erf}\zeta/\operatorname{erf}\alpha, \tag{12}$$

$$\Theta_2 = \operatorname{erfc}(\zeta/\sqrt{\varkappa})/\operatorname{erfc}(\alpha\sqrt{\varkappa}), \tag{13}$$

$$\alpha\,\text{Ko} = \frac{\exp(-\alpha^2)}{\sqrt{\pi}\operatorname{erf}\alpha} - \text{K}_\text{T}\text{Ke}\frac{\exp(-\alpha^2/\varkappa)}{\sqrt{\pi}\operatorname{erfc}(\alpha/\sqrt{\varkappa})}. \tag{14}$$

The Leibenzon–Charnyi method yields ($\alpha = \alpha_1$):

$$\Theta_1 = \zeta/\alpha,\ \Theta_2 = \operatorname{erfc}\frac{\zeta-\alpha}{\sqrt{\varkappa}}, \tag{15}$$

$$\alpha\,\text{Ko} = \frac{1}{2\alpha} - \frac{\text{K}_\text{T}\,\text{Ke}}{\sqrt{\pi}}. \tag{16}$$

The integral method ($\alpha = \alpha_2$, $\mu = \mu_2$):

$$\Theta_1 = \zeta/\alpha,\ \Theta_2 = \left(\frac{\zeta/\alpha-\mu}{1-\mu}\right)^2, \tag{17}$$

$$\alpha\,\text{Ko} = \frac{1}{2\alpha} - \frac{\lambda\,\text{K}_\text{T}}{\alpha(\mu-1)}. \tag{18}$$

Here we find an additional parameter which associates the laws governing the motion of the phase separation boundary S(t) and the thermal effect zone R(t). In this case, for the determination of R(t) two conditions are given [3, 4]:

$$\Theta_2 = 0,\ \partial\Theta_2/\partial x = 0 \text{ for } x = R(t). \tag{19}$$

In the case of a self-similar solution [8]:

$$R(t) = \mu S(t). \tag{20}$$

The parameter $\mu$ is found from the solution of the quadratic equation

$$(\mu + 2)(\mu - 1 - 2\lambda\,\mathrm{K\tau}) - 6\varkappa\,\mathrm{Ko} = 0. \tag{21}$$

Let us note that this parameter depends significantly on the initial data [8], whereas in [9] it is assumed to be equal to 4.5-5.0.

The successive-approximation method ($\alpha = \alpha_3$, $\mu = \mu_3$):

$$\Theta_1 = \zeta/\alpha + \zeta\frac{\alpha}{3}[1 - (\zeta/\alpha)^2], \tag{22}$$

$$\Theta_2 = \left[1 + \frac{\alpha^2 + \alpha\mu(\alpha - \zeta) - \zeta^2}{3\varkappa}\right]\frac{\mu - \zeta/\alpha}{\mu - 1}, \tag{23}$$

$$\alpha\,\mathrm{Ko} = \frac{1}{2\alpha} - \frac{\alpha}{3} - \lambda\,\mathrm{K\tau}\left[\frac{1}{2\alpha(\mu - 1)} + \frac{\alpha(\mu + 2)}{6\varkappa}\right], \tag{24}$$

$$(2\mu + 1)(\mu - 1 - \lambda\,\mathrm{K\tau}) - \lambda\,\mathrm{K\tau}(\mu + 2) - 6\varkappa\left(\mathrm{Ko} + \frac{1}{3}\right) = 0. \tag{25}$$

With the combination method ($\alpha = \alpha_4$, $\mu = \mu_4$) the temperature distribution in the first zone is found by the method of successive approximations, while in the second zone it is found by the integral method. As a result, the solution of the problem is given by formula (22) and the second of the formulas in (17), as well as by the relationships

$$\alpha\,\mathrm{Ko} = \frac{1}{2\alpha} - \frac{\alpha}{3} - \frac{\lambda\,\mathrm{K\tau}}{\alpha(\mu - 1)}, \tag{26}$$

$$(\mu + 2)(\mu - 1 - 2\lambda\,\mathrm{K\tau}) - 6\varkappa\left(\mathrm{Ko} + \frac{1}{3}\right) = 0. \tag{27}$$

<u>2. Solution of the Problem for Boundary Condition of the 2nd Kind (6).</u> In this case the temperature distribution in the second zone retains its original form; however, the sense of the coefficients $\alpha$ and $\mu$ which are contained in formulas (13), (15), (17), and (23) changes.

The exact self-similar solution

$$\bar{\Theta}_1 = \frac{\sqrt{\pi}}{2}(\mathrm{erf}\,\alpha - \mathrm{erf}\,\zeta), \tag{28}$$

$$\alpha\,\overline{\mathrm{Ko}} = \frac{\exp(-\alpha^2)}{2} - \frac{\overline{\mathrm{K\tau}}\,\mathrm{Ke}}{\sqrt{\pi}}\,\frac{\exp(-\alpha^2/\varkappa)}{\mathrm{erfc}(\alpha/\sqrt{\varkappa})}. \tag{29}$$

The Leibenzon–Charnyi method:

$$\bar{\Theta}_1 = \alpha - \zeta, \tag{30}$$

$$\alpha\,\overline{\mathrm{Ko}} = \frac{1}{2} - \frac{\overline{\mathrm{K\tau}}\,\mathrm{Ke}}{\sqrt{\pi}}. \tag{31}$$

The integral method:

$$\alpha\,\overline{\mathrm{Ko}} = \frac{1}{2} - \frac{\lambda\overline{\mathrm{K\tau}}}{\alpha(\mu - 1)}. \tag{32}$$

The temperature distribution in the first zone is given by formula (30), while the parameter $\mu$ is given by (21).

The successive-approximation method:

$$\bar{\Theta}_1 = \alpha - \zeta + \frac{\zeta^3 - \alpha^3}{3}, \tag{33}$$

TABLE 1. Comparison of Calculation Methods for Various Boundary Conditions

| Method | $\alpha$ | | $\mid \alpha - \alpha_T \mid / \alpha_T \cdot 100\%$ | |
|---|---|---|---|---|
| | boundary conditions | | | |
| | 1st kind | 2nd kind | 1st kind | 2nd kind |
| LChM | 0,0800 | 0,2476 | 6,59 | 9,28 |
| IM | 0,0747 | 0,2409 | 0,51 | 6,29 |
| SAM | 0,0701 | 0,2239 | 6,69 | 1,19 |
| CM | 0,0745 | 0,2267 | 0,85 | 0,55 |
| Exact solution | 0,0751 | 0,2266 | 0 | 0 |

TABLE 2. Distribution of $1 - \Theta_1(\zeta)$ in Thawing Zone for Boundary Condition of the 1-st Kind

| Method, $\zeta/\alpha_T$ | 0,02 | 0,20 | 0,60 | 0,92 | 1 |
|---|---|---|---|---|---|
| LChM | 0,981 | 0,812 | 0,437 | 0,137 | 0,62 |
| IM | 0,980 | 0,800 | 0,397 | 0,075 | — |
| SAM | 0,978 | 0,785 | 0,359 | 0,014 | — |
| CM | 0,980 | 0,800 | 0,394 | 0,072 | — |
| Exact solution | 0,980 | 0,800 | 0,399 | 0,080 | 0 |

TABLE 3. Distribution of $\Theta_2(\zeta) - 1$ in Frozen Zone for Boundary Condition of the 1-st Kind

| Methods, $\zeta/\alpha_T$ | 1 | 1,58 | 6,82 | 21,9 | 30,1 |
|---|---|---|---|---|---|
| LChM | — | —0,033 | —0,355 | —0,906 | —0,980 |
| IM | — | —0,040 | —0,360 | —0,922 | —1,000 |
| SAM | —0,005 | —0,047 | —0,415 | —1,000 | —1,000 |
| CM | — | —0,040 | —0,361 | —0,922 | —1 000 |
| Exact solution | — | —0,039 | —0,375 | —0,916 | —0,983 |

TABLE 4. Values of the Function $\Theta_2(\zeta)/\Theta_{1T})(0)$ for Boundary Condition of the 2-nd Kind

| Methods, $\zeta/\alpha_T$ | 0 | 0,20 | 0,60 | 0,80 | 0,98 | 1,00 |
|---|---|---|---|---|---|---|
| LChM | 1,112 | 0,908 | 0,501 | 0,298 | 0,115 | 0,094 |
| IM | 1,081 | 0,878 | 0,471 | 0,267 | 0,084 | 0,064 |
| SAM | 0,988 | 0,785 | 0,382 | 0,183 | 0,008 | — |
| CM | 1,000 | 0,797 | 0,394 | 0,195 | 0,020 | 0,000 |
| Exact solution | 1,000 | 0,797 | 0,393 | 0,195 | 0,019 | 0 |

TABLE 5. Values of the Function $\Theta_2(\zeta) - 1$ for Boundary Condition of the 2-nd Kind

| Methods, $\zeta/\alpha_T$ | 1 | 1,17 | 2,74 | 4,49 | 7,45 | 9,72 |
|---|---|---|---|---|---|---|
| LChM | — | —0,016 | —0,310 | —0,589 | —0,876 | —0,963 |
| IM | — | —0,025 | —0,351 | —0,635 | —0,931 | —1,000 |
| SAM | —0,003 | —0,046 | —0,411 | —0,744 | —1,000 | —1,000 |
| CM | — | —0,040 | —0,363 | —0,645 | —0,936 | —1,000 |
| Exact solution | — | —0,040 | —0,373 | —0,657 | —0,919 | —0,977 |

$$\alpha \bar{Ko} = \frac{1-\alpha^2}{2} - \lambda \bar{Kt} \left[ \frac{1}{2\alpha(\mu-1)} + \frac{\alpha(\mu+2)}{6\varkappa} \right]. \quad (34)$$

Here $\mu$ is determined from Eq. (25).

The combination method:

$$\alpha \bar{Ko} = \frac{1-\alpha^2}{2} - \frac{\lambda \bar{Kt}}{\alpha(\mu-1)}. \quad (35)$$

Here $\bar{\Theta}_1$ coincides with (33), and $\mu$ is found from Eq. (27).

Let us note that the transcendental equation (14) has positive roots for all values of the initial parameters, whereas in the case of Eq. (29) the roots are positive only when the following inequality is satisfied:

$$\bar{Kt}\,Ke < \sqrt{\pi}/2. \quad (36)$$

It is interesting that it is precisely such an inequality that follows from formula (31). Other approximate solutions bring us to somewhat different limitations on the magnitudes of the initial parameters.

A quantitative comparison of the considered approximation methods relative to the exact solutions was carried out for the case of the thawing of the frozen soil in the case of the following values of the dimensionless complexes: $\lambda = 1.433$, $\varkappa = 1.755$, Ke = 1.082, Ko = $\bar{Ko}$ = 1.722, Kt = 10, $\bar{Kt}$ = 0.1. The results of the calculations are presented in Tables 1-5. We see that in the determination of the law governing the motion of the phase-separation boundary the Leibenzon—Charnyi method always majorizes the exact solution (see Table 1); when the phase-separation boundary moves slowly it is the integral method that is most exact (see the second and fourth columns of Table 1); however, with increasing velocity it falls behind the combination method and even behind the successive-approximation method (the third and fifth columns in Table 1); in ths calculations for the temperature distributions in the thawing zone, it is the combination method that is most exact (see Tables 2 and 4), while for the frozen zone this is valid only for the initial segments of the temperature curve (see Tables 3 and 5).

## NOTATION

t, time; x, a coordinate; T, temperature; Q, specific heat flow (W/m); $\lambda_i$, $c_i$, $a_i$, thermal conductivity, the volumetric heat capacity, and thermal diffusivity of the thawing (i = 1) and frozen (i = 2) zones; $\ell$, latent heat of phase transition between ice and water; $\omega$, moisture of the soil (in fractions of unity); $\rho$, density; S(t), R(t), movable boundaries of the phase transition and the thermal effect; $Ke = \sqrt{\lambda_2 c_2/\lambda_1 c_1}$; $\lambda = \lambda_2/\lambda_1$; $\varkappa = a_2/a_1$; $Ko = \ell\rho\omega/c_1(T_c - T_{ph})$, Kossovich number; $Kt = (T_{ph} - T_0)/(T_c - T_{ph}) > 0$; $\Theta_1 = (T_1 - T_c)/(T_{ph} - T_c)$; $\Theta_2 = (T_2 - T_0)/(T_{ph} - T_0)$; $\bar{Ko} = \ell\rho\omega/(Q/a_1)$; $\bar{Kt} = (T_{ph} - T_0)/(Q/\lambda_1)$; $\bar{\Theta}_1 = (T_1 - T_{ph})/(Q/\lambda_1)$.

## LITERATURE CITED


1. M. S. Leibenzon, Management of Petroleum Industry Mechanics. Issue 41. Hydraulics. [in Russian], Moscow—Leningrad (1931).
2. I. A. Charnyi, Sovetskii metropoliten, No. 4, 29-34 (1940).
3. G. I. Barenblatt, Izv. Akad. Nauk SSSR, Ser. Mekh. Mashinostr., No. 9, 35-49 (1954).
4. J. Goodman, Trans. ASME, 80, 335-342 (1959).
5. F. Kreith and F. F. Romie, Proc. Phys. Soc., Sec. B, 68, 277-291 (1955).
6. M. S. El-Genk and A. W. Gronenberg, Letters on Heat and Mass Transfer, 6, 321-327 (1979).
7. G. Carslaw and D. Eger, Heat Conduction in Solids [Russian translation], Moscow (1964).
8. É. A. Bondarev, T. A. Kapitonova, and F. S. Popov, Thermophysical and Mass Transfer Properties of Hygroscopic Materials [in Russian], Yakutsk (1977), pp. 83-92.
9. Kh. R. Khakimov, Problems in the Theory and Practice of Artificial Soil Freezing [in Russian], Moscow (1957).